Everyone Math Algebraic Expression Series-3, Solving equation series-8 (not done yet. Need AI revise 7/14/2025)

Why $\sqrt[3]{-1}$ is a bad notation (but people keep using it)?

3rd degree equations -2

Meijun Zhu

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What is $\sqrt[3]{-1}$? The answer depends on whether you understand complex numbers or not. It will be a challenging question for the computer (i. e. AI) since you need to give the machine more conditions. We thus claim the notation is a bad one.

As we discussed before, $\sqrt{-15}$ is not a good notation, in particular, if you view that

$$\sqrt{-15} = \sqrt{15} \times \sqrt{-1} = \sqrt{15}i$$

follows from the exponent rule:

$$(ab)^r = a^r b^r,$$

since this rule fails for negative numbers a, b when r is a fractional exponent.

It is okay, if you insist on defining

$$\sqrt{-x} = \sqrt{x}i$$

for any positive number x. But we prefer to use $\sqrt{x}i$ directly.

Now coming to $\sqrt[3]{-1}$, it is more subtle to convince people of abandoning this notation since we were taught in young age that $\sqrt[3]{-1} = -1$! For example, if we are solving equation:

$$x^3 = -1,$$

We know $x = -1$ is one solution. Other solutions? Some people will write $x = \sqrt[3]{-1}$ to represent three solutions! Unfortunately, this is unacceptable: as we pointed out in Algebraic expression series -1, the algebraic expression $\sqrt[3]{-1}$ can only represent one value.

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Why do people keep using the ambiguous notation $\sqrt[3]{-1}$? It is quite similar to the notation $\sqrt{-15}$: if you use quadratic formula without thinking (yes, even very popular GPT may use the formula without blinking its "eyes"), you will write the notations like $\sqrt{-15}$ often.

For a general 3rd degree polynomial equation

$$x^3 + rx^2 + sx + t = 0,$$

we can always use substitution $y = x - c$ for suitable c (you may argue this abstractly using Taylor series) to get an equivalent equation

$$y^3 + py + q = 0. \tag{1}$$

Is there a uniform "formula" that we can use to write down all solutions to equation (1)? Yes, there is one, due to Cardano, Tartaglia and Dal Ferro (here for convenience, I just give the credit to the original inventor: Dal Ferro around 1500):

The Dal Ferror's formula:

$$\boldsymbol{y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}} \tag{2}$$

Let us try to solve

$$y^3 - 3y + 2 = 0.$$

Using Dal Ferro's formula, we have

$$y = \sqrt[3]{-1} + \sqrt[3]{-1}.$$

If you use the "traditional understanding: $\sqrt[3]{-1} = -1$, you get one solution: $y_1 = -2$. But the formula fails to yield two other solutions: $y_2 = y_3 = 1$! So, we need to think about: what is $\sqrt[3]{-1}$?

Similar to how we derive the quadratic formula, we exam how to obtain formula (2).

The cute trick for solving equation (1) is to introduce two new variables u and v, so that $y = u + v$. Bringing it to equation (1), we have

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0. \tag{3}$$

If u and v satisfy

$$\begin{cases} u^3 + v^3 = -q \\ uv = -\dfrac{p}{3} \end{cases} \tag{4}$$

$y = u + v$ will be a solution to equation (1). Usually, it is not easy to solve high order system like system (4). However, we observe: if u and v satisfy system (4), they also satisfy

$$\begin{cases} u^3 + v^3 = -q \\ u^3 v^3 = -\dfrac{p^3}{27} \end{cases} \tag{5}$$

System (5) can be solved by (yes, using the quadratic formula):

$$\begin{cases} u^3 = -\dfrac{q}{2} \pm \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}} \\ v^3 = -\dfrac{q}{2} \mp \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}} \end{cases} \tag{6}$$

Once we take cube roots (if we can?), we will get formula (2).

There are two drawbacks in the above derivation.

(i). If $\frac{q^2}{4} + \frac{p^3}{27} < 0$, as we discussed before, we shall replace $\sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$ by $i\sqrt{\left|\frac{q^2}{4} + \frac{p^3}{27}\right|}$

(ii). We shall NOT take a cube root to derive (2) from (6), rather, we need to find three cube roots from (6).

Example 1: Solve the equation

$$y^3 - 3y + 2 = 0.$$

Solution: Let $y = u + v$, where u and v satisfy (we use formula (5) here):

$$\begin{cases} u^3 + v^3 = -2 \\ u^3 v^3 = 1 \end{cases}$$

Thus, we obtain

$$\begin{cases} u^3 = -1 \\ v^3 = -1 \end{cases}$$

The three cube roots of -1 are:

$$\frac{1}{2}+\frac{\sqrt{3}}{2}i, \qquad -1, \qquad and \qquad \frac{1}{2}-\frac{\sqrt{3}}{2}i.$$

We then can verify that when $u = v = -1$, or $u = \frac{1}{2}+\frac{\sqrt{3}}{2}i \ \ and \ \ v = \frac{1}{2}-\frac{\sqrt{3}}{2}i$, or $v = \frac{1}{2}+\frac{\sqrt{3}}{2}i \ \ and \ \ u = \frac{1}{2}-\frac{\sqrt{3}}{2}i$, we have three solutions: $y_1 = -2$, $y_2 = 1, y_3 = 1$.

Summary

Even we have quadratic formula for quadratic equation, and Dal Ferror's formula for the 3rd degree equation, we can not direct use them. Rather, we need to understand the real meaning of the square root and cube root in these two formulas!